Divisibility: A Problem Solving Approach Through Generalizing and Specializing

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ABSTRACT
This paper describes a divisibility rule for any prime number as an engaging problem solving activity for preservice secondary school mathematics teachers.

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My students, preservice secondary school mathematics teachers holding majors or minors in mathematics or science, were raised to believe that there were some "neat" divisibility rules for numbers like 2, 3, 5, 9, 10, 100, some considering the last digit or digits and some considering the sum of the digits. They have also heard of some "weird" and totally unuseful divisibility rules for 7, 11 and maybe even 13. Usually, the former are introduced, and at times even proved, in junior high school. The latter are mentioned briefly without a proof, or omitted altogether. In the AS (After Sputnik) era of growing dependence on calculating machines, who could possibly be interested in divisibility rules?

The curiosity of one student generated an interesting investigation that I wish to present here. This student discovered, in fact found on the internet, a divisibility rule for 7, and wondered why it worked. I blessed her curiosity and suggested that the class work on it. The results went far beyond our original intentions: a divisibility rule for any prime number has been derived and proved. More than the mathematical exercise, I wish to share the exciting mathematical investigation and experimentation in which the students engaged.

I will present the results as a problem solving activity that started with collecting data through observation and incorporated several rounds of implementing a "What if not?" strategy (Brown & Walter, 1990). I will present the results as students' engagement in generalizing and specializing (Mason, 1985) and will conclude with a brief discussion on the relevance of such an activity as well as several ideas for possible extensions.

PROLOGUE

Consider the divisibility rule for 3: A number is divisible by 3 if and only if the sum of its digits is divisible by 3. Let's prove it for a 4 digit number.

Consider an expanded notation of a 4 digit number written with digits a,b,c and d from left to right.

$$1000a + 100b + 10c + d = (999a + a) + (99b + b) + (9c + c) + d$$

Applying associativity and commutativity of addition, this equals

$$(999a + 99b + 9c) + (a+b+c+d)$$

The first addend in this sum $(999a + 99b + 9c)$ is always divisible by 3. The second addend $(a+b+c+d)$ is the sum of the digits. Therefore, the number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Even though this proof refers to a four digit number, it gives a general idea how the proof can be extended to a number with n-digits. The strategy used in this proof is representing a number as a sum of two addends. The divisibility of one component is obvious. The divisibility of the second component determines the divisibility of the number. A similar strategy will be applied in the following proofs.

DIVISIBILITY BY 7 - INTRODUCING THE ALGORITHM

Divisibility of a number by 7 can be determined using the following recursive algorithm:

1) Multiply the last digit of the number by 2.
2) Subtract the product in (1) from the number obtained by deleting the last digit of the original number.
3) Continue steps 1 and 2 until the divisibility of number obtained in (2) by 7 is "obvious." The original number is divisible by 7 if and only if the number obtained in step (2) is divisible by 7.

EXAMPLES OF IMPLEMENTATION:
(a) Is 86415 divisible by 7?
86415 —> 8641 - (5x2) = 8631
8631 —> 863 - (1x2) = 861
861 —> 86 - (1x2)= 84
84 —> 8 - (4x2) = 0
Yes, 0 is divisible by 7, therefore 86415 is divisible by 7.

(b) Is 380247 divisible by 7?
380247 —> 38024 - (7x2) = 38010
38010 —> 3801 - (0x2) = 3801
3801 —> 380 - (lx2) = 378
378 —> 37 - (8x2) = 21
21 is divisible by 7, and, therefore, 380247 is divisible by 7. (We could continue one step further to get a zero).

(c) Is 380245 divisible by 7?
380245 —> 38024 - (5x2) = 38014
38014 —> 3801 - (4x2) = 3793
3793 —> 379 - (3x2) = 373
373 —> 37 - (3x2) = 31
31 is not divisible by 7, and, therefore, 380247 is not divisible by 7.

When examples similar to the above were presented in class, the immediate response for many students was a desire to try it out, to carry out the algorithm on numbers of their choice and verify divisibility with a calculator. This generated a large body of evidence to suggest that the algorithm "works." This also generated two related questions:

1) Why does this work?
2) Why does this work for 7?

The first question is drawn by the desire to understand the algorithm and to prove that it determines divisibility by 7 for any natural (or integer) number. The second question is drawn by the desire to determine the special place of the number 7 in the algorithm. Specializing on 7 in turn invites generalization: Does it work for 7 only? Will the algorithm work for another number? For which numbers will it work? How can the algorithm be modified to work for another number?

WHAT IF NOT 7?
While experimenting with other numbers, a lucky trial by one student prompted a conjecture, that exactly the same algorithm can be applied to determine divisibility by 3. This conjecture has been supported by several examples, however, no other number was found for which the above algorithm can be applied to determine divisibility. To encourage further investigation I suggested the following variation.

DIVISIBILITY BY 19 - VARYING THE ALGORITHM
Divisibility of a number by 19 can be determined by the following algorithm:

1) Multiply the last digit of the number by 2.
2) Add the product in (1) to the number obtained by deleting the last digit of the original number.
3. Continue steps 1 and 2 till the divisibility of number obtained in (2) by 19 is "obvious." The original number is divisible by 19 if and only if the number obtained in step (2) is divisible by 19.

EXAMPLES OF IMPLEMENTATION
(a) Is 15276 divisible by 19?
15276 —> 1527 + (6x2) = 1539
1539 —> 153 + (9x2) = 171
171 —> 17 + (1x2) = 19
19 is divisible by 19, and, therefore, 15276 is divisible by 19.

(b) Is 12312 divisible by 19?
12312 —> 1231 + (2x2) = 1235
1235 —> 123 + (5x2) = 133
133 —> 13 + (3x2) = 19
19 is divisible by 19, and, therefore, 12312 is divisible by 19.

For convenience of reference in further discussion, we shall name this algorithm a *trimming algorithm*.

WHY-QUESTIONS TO PONDER
Experimenting with the two variations of the trim-

ming algorithm presented above there are (at least) two questions that arise:

1) Why is the last digit multiplied by 2?
2) Why does the algorithm involve subtraction in case of 7 and addition in case of 19?

DIVISIBILITY BY 17 - ANOTHER VARIATION

A different variation on the trimming algorithm can be used to determine divisibility by 17. In this case we multiply the last digit by 5 and subtract the product from the "trimmed" number:

EXAMPLES:

(a) Is 82654 divisible by 17?
82654 —> 8265 - (4x5) = 8245
8245 —> 824 - (5x5) = 799
799 —> 79 - (9x5) = 34
we may stop here or continue one step further
34 —> 3 - (4x5) = -17
Conclusion: 82654 is divisible by 17.

(b) Is 17456 divisible by 17?
17456 — > 1745 - (6x5) = 1715
1715 — > 171 - (5x5) = 146
146 — > 14 - (6x5) = -16
Conclusion: 17456 is not divisible by 17.

REPHRASING THE WHY-QUESTIONS

The similarities among the three algorithms are obvious. However, the last variation suggests rewording of the first question:

1) How is the multiplier of the last digit of the number determined? (Why was it 2 in case of 19 and 7 and 5 in case of 17?)

The second question remains basically the same:

2) Why does the algorithm involve addition in some cases and subtraction is others?

DIVISIBILITY BY 7 - A SPECIFIC "GENERIC" PROOF

After experimenting with a variety of examples the students became convinced that the algorithms do indeed represent a divisibility rule. However, they were still seen as some magic tricks. The interest in WHY (they work) took over from the initial excitement of HOW they work.

Let us prove the divisibility algorithm for 7.

Consider any natural number n. If N is the number obtained from n by deleting the last digit a, we can always represent n as $10N+a$. (Example: 3456 = 10 x 345 + 6) We are interested in connecting our original number n and the number obtained by the algorithm, namely, N-2a. In fact, we would like to prove that n is divisible by 7 if and only if N-2a is divisible by 7.

Applying simple arithmetic we get:

$$\begin{aligned} 10N + a &= 10(N - 2a) + 20a + a \\ &= 10(N - 2a) + 21a \end{aligned}$$

The last addend (21a) is divisible by 7 for any digit a. Therefore n is divisible by 7 if and only if N-2a is divisible by 7. Now we can treat the "new" number (N-2a) as the number for which divisibility by 7 has to be established using the same method.

WHAT IF NOT 7?

What if divisibility by a prime number p is in question? Separate proofs, similar to the above, can be developed for a variety of numbers. Inviting students to develop these proofs and discuss similarities among them may help in generalizing to attain an algorithm which determines divisibility of a number by any prime p.

DIVISIBILITY BY *P* - GENERALIZING THE ALGORITHM

In order to construct an algorithm to determnine divisibility by a prime number p we are looking for a natural number k such that $10k\pm1$ is divisible by p. Then,

$$\begin{aligned} 10N + a &= 10(N \mp ka) \pm 10ka + a \\ &= 10(N \mp ka) \pm (10k \pm 1)\, a \end{aligned}$$

If $10k\pm1$ is divisible by p, then $(10k\pm1)a$ is divisible by p for any digit a. Therefore, $10N+a$ (which is our number n) is divisible by p if and only if $N \mp ka$ (the number obtained by applying the algorithm) is divisible by p.

SPECIALIZING: DIVISIBILITY BY 17

For example, to determine divisibility by 17 we looked for a number of the form $10k\pm1$ divisible by 17. We found 51. Therefore, k=5. This is the number used in the trimming algorithm to establish divisibility for 17. Since 51 has the form $10k+1$, the number obtained in the algorithm should be of the form N-ka, therefore the algorithm involves a subtraction of a product of the last digit by 5.

SPECIALIZING: DIVISIBILITY BY 31

What is divisibility rule for 31? 31 itself differs by 1 from the closest multiple of 10. Therefore, k=3 and the algorithm involves subtraction.

EXAMPLE:

Is 4185 divisible by 31?
4185 —> 418 - (50) = 403
403 —> 40 - (30) = 31
Conclusion: Indeed, 4185 is divisible by 31.

SPECIALIZING: DIVISIBILITY BY 13

What is the divisibility rule for 13? We find 39 as a multiple of 13 that differs by 1 from a multiple closest to 10. Therefore k=4 and the algorithm involves addition.

EXAMPLE:

Is 4173 divisible by 13?
4173 —> 417 + (3x4) = 429
429 —> 42 + (9x4) = 78
78 —> 7 + (8x4) = 39
Conclusion: 4173 is divisible by 13.

EXISTENCE PROOF

We believe that so far the why questions (1) and (2) raised earlier have been answered. Now it is time to wonder whether it is possible to find an appropriate trimming algorithm to determine divisibility by any prime.

The mathematical answer is no. However, the "human" answer is--almost. Such an algorithm can be determined for all the primes except 2 and 5. (However, since 2 and 5 have well known divisibility rules, we will focus on the other primes.)

The existence of a trimming algorithm for p depends on the existence of a multiple of p which is larger by 1 or smaller by 1 than a multiple of 10. It is obvious that such a multiple does not exist for 5 and 2 which are factors of 10. Let us prove its existence for all other primes.

Formally, let's prove that for any prime p, $p \neq 2, 5$, there exist natural numbers k and m such that $|mp-10k|=1$.

Let us consider the last digit x of p. The possibilities are 1,3,7 or 9, since this digit cannot be even or 5. If x=1 or x=9 the prime itself differs by 1 from the closest multiple of 10. In this case m=1 and k is determined accordingly. If x=3, let m=3, then the last digit of mp is 9 and the number mp is smaller than the closest multiple of 10 by 1. If x=7 , let m=3, then the last digit of mp is 1 and the number mp is bigger than the closest multiple of 10 by 1. Therefore if x=3 or x=7, then m=3 and k is determined accordingly.

In summary, for any prime p, $p \neq 2, 5$, it is possible to determine a divisibility rule based on a trimming algorithm.

NUMBER THEORY CONNECTION

In a number theory text (e.g. Long, 1987, p. 98) the following can be found as an exercise:

(a) If p is a prime and $(p, 10)=1$, prove that there exist integers k and y such that $yp=10k+1$.
(b) Let $n=10a+b$. If p is a prime with $(p, 10)=1$, prove that p/n if and only if $p/(a-kb)$, where k is determined in (a).

However, without a concrete experience the relationship between this exercise and divisibility rules may not be apparent to many students.

FOR FURTHER INVESTIGATION

In Polya's tradition, the fourth step in problem solving is "looking back" (Polya, 1988). This involves searching for alternative solutions or solution paths, generalizing solutions and exploring situations to which the problem or the method of solution can be applied. We have presented one level of looking back at the problem of divisibility by 7 by exploring the divisibility algorithm for any prime. Further, "looking back" at the general divisibility rule, we can extend our investigation by asking several "what if not" questions.

- What if not primes? Can a similar algorithm be used or modified to determine divisibility by a composite number? What properties of primes were used in our proof? For what composite numbers can the algorithm be applied or modified?
- In all the above examples we have chosen the smallest multiple of p that was bigger by 1 or smaller by 1 than a multiple of 10. However, in our proof there was no reference to the choice of the smallest k. So, what if not the smallest? Will

the algorithm still work? In other words, is the algorithm, for which existence is proved above, unique?

- Which familiar divisibility rules can be seen as special cases of the general algorithm?

A COMMENT ON USEFULNESS

There is a common trend in mathematics education to focus on "applicable" mathematics, on mathematics that is related to "real life situations." From this perspective, there is a danger of labeling divisibility rules as "unuseful."

I believe that usefulness, together with mathematical beauty, is in the eye of the beholder. For me, a problem that attracts students' interest and curiosity, that generates an engaging investigation, that invites students to make conjectures and test conjecture--is most useful. I believe that an engaging mathematical investigation is useful for all learners, and the excitement of mathematical investigation is especially useful for individuals planning for a teaching career. I hope that my students feel the same.

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Excerpts from Ivan's Commandments for Himself

Ivan Niven

Ivan Niven of the University of Oregon, author of Mathematics of Choice and several texts on number theory, died on May 9, 1999, at age 83. At a memorial service in his honor, the program included the item below, which was found among his personal papers.

Thou shalt make an unceasing effort to see the world as it truly is, not as a product of your desires, not as a work of your imagination, not as a matrix of your special interests, but as an external reality that is no respecter of persons.

Thou shalt not deliberately misstate or misrepresent another's position by exaggeration, by quotation out of context or by confusing a statement and its converse. Neither shall thou attempt to destroy another's position by harping on some error or minor defect that in no way affects his principal contention.

Thou shalt not claim to know more than thou knowest. Thou shalt judge the merits of a proposal in terms of its own worth, irrespective of the proponents thereof.

Thou shalt not exalt trivial matters, nor claim as primary what is at best secondary. For who but a foolish person will resign from his church, his political party or his club because of one or two speeches or occurrences not to his liking.

Thou shalt have the grace to concede a point without going into a huff, without claiming that wasn't what you said, or meant to say, and without saying, "Didn't you know I was only kidding?"

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